The force on a sphere moving through a conducting fluid in the presence of a magnetic field

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The force on a sphere moving through an inviscid, conducting fluid in the presence of a uniform magnetic field $\mathbf{B}_0$ is calculated for the low-conductivity case where the hydrodynamic motion deviates only slightly from potential flow. The magnetic Reynolds number is assumed small. The force on the sphere is found to consist of both a drag and a deflective component which tends to orient its motion parallel to a magnetic field line; if the sphere's velocity is $\mathbf{V}$, the force may be written

$$\mathbf{R} = -AB_0^2\mathbf{V} + C(\mathbf{V}.\mathbf{B}_0)\,\mathbf{B}_0,$$

where the coefficients A and C depend on the conductivities of both sphere and fluid. The coefficients are evaluated by calculating the Joule dissipation for particular orientations of $\mathbf{V}$ relative to $\mathbf{B}_0$. In one case the force is also calculated directly from the perturbed pressure distribution in the fluid. In an analogous way, a spinning sphere in a conducting fluid experiences both resistive and gyroscopic torques.

1. Introduction

An uncharged, non-magnetic body moving through an inviscid conducting fluid or plasma in the presence of a magnetic field should experience a resistive drag force or torque due to the Joule dissipation of energy caused by induced currents in the conducting system. We call this *induction drag*. Accompanying such drag forces are generally deflective forces or torques of similar magnitude at right angles to the velocity or angular velocity of the body, which might be called induction deflexion. The problem in the case of a strong magnetic field has been studied in some detail by Stewartson (1956). For the case of a weak magnetic field, calculations of the induction drag for a sphere have been made by Chopra (1956, 1957) and by Chopra & Singer (1958), but the expressions they obtain are incorrect as is evident from the fact that their drag force does not vanish when the conductivity of the fluid goes to zero, a result in conflict with the Special Theory of Relativity. Their error appears to arise, among other things, from a failure to observe that a concomitant of such motion is the induction of charges on the surface of a sphere (and in the case of a spinning sphere, induced volume charge density in the sphere itself) producing electric fields of comparable magnitude to those produced by induction (dynamo effect). There are additional effects if the body is charged or magnetized, but such cases are not considered in this paper.

It is the purpose of the present paper to provide correct expressions for induction drag and deflective forces for a sphere moving with constant velocity, and expressions for induction drag and deflective torques for a spinning sphere, on the assumption that the induced magnetic field is small and that the hydrodynamic motion differs only slightly from the situation of potential flow. Such situations should exist when the magnetic field is sufficiently weak or the conductivity of the fluid is sufficiently small. In obtaining drag and deflective forces it is tacitly assumed that the fluid motion is steady in the system of co-ordinates in which the sphere is at rest. We have not been able to show that our solution to the problem is unique or even that a steady-state solution of the magnetohydrodynamic equations exists in all cases of this type. However, for the case of motion of the sphere parallel to the magnetic lines of induction, we demonstrate that the drag force obtained from Joule losses is the same as that computed from the perturbed pressure distribution in the fluid to first order in the fluid conductivity.† The assumption that the zero-order magnetic field can be taken to be a uniform field in these cases also requires examination, since Ludford & Murray (1960) have shown that for a similar geometry, the perturbation expansion of the magnetic field in the fluid conductivity is not regular.

In §2 we carry out in a straightforward manner the calculation of the forces by the use of the Joule-loss method for several cases of interest. In §3, we examine in more detail the assumptions and approximations made in the special case of a sphere moving parallel to the magnetic field and verify that a complete first-order calculation yields the same result for the drag force as does the Joule-loss calculation.

2. Force and torque from Joule losses

Consider a homogeneous spherical body of radius a and electrical conductivity σ' moving with velocity $-\mathbf{V}$ (or spinning with angular velocity $\boldsymbol{\omega}$) in an inviscid, incompressible fluid of conductivity σ in the presence of a uniform magnetic field $\mathbf{B}_0$. The Joule losses resulting from the induced currents in the sphere-fluid system will be calculated and equated to the rate at which mechanical energy is dissipated by the sphere, $\mathbf{R}.\mathbf{V}$ (or $\mathbf{D}.\boldsymbol{\omega}$). These calculations enable us to obtain the drag force, $\mathbf{R}$, or the drag torque, $\mathbf{D}$, respectively.

In calculating the induced currents, the magnetic field is taken to be the zero-order uniform field, and the fluid velocity is taken to be that which would exist if the conductivity of the system were zero. The first of these statements requires that the magnetic Reynolds number, $R_M = \sigma\mu Va$, be small and that the perturbation expansion of the magnetic field in terms of this parameter, although irregular at large distances, does not change the uniform field in the current-carrying region. The second statement requires that the parameter $\beta R_M = \sigma a B_0^2/\rho V$ be small and that the perturbation expansion for the fluid velocity be sufficiently regular in the same sense. These remarks will be amplified in §§3.2 and 3.3. Here ρ is the density of the fluid.

† After completion of this work, our attention has been directed to a paper by Ludford (1960) who also makes the point that, for flow at low magnetic Reynolds number, the induction drag can be computed directly from the undisturbed potential flow and magnetic field.

No restrictions will be placed on the conductivity σ' of the translating sphere,† although for the rotating sphere it will be necessary to assume that R'_M (defined using σ' and μ') is small also. We shall take the permeabilities of sphere and fluid equal to each other so that induced dipole effects may be neglected.

2.1. *Force on a moving sphere from Joule losses*

The basic equations of the electromagnetic field under steady conditions are:

$$(a)\ \operatorname{div}\mathbf{B} = 0,\quad (b)\operatorname{curl}\mathbf{E} = 0,\quad (c)\operatorname{curl}\mathbf{B} = \mu\mathbf{j},\quad (d)\ \operatorname{div}\mathbf{E} = \rho/\epsilon,$$
$$(e)\operatorname{div}\mathbf{j} = 0,\quad (f)\ \mathbf{j} = \sigma\mathscr{E}, \tag{1}$$

where, for convenience, we have defined $\mathscr{E}$ by

$$\mathscr{E} = \mathbf{E}+\mathbf{v}\times\mathbf{B}, \tag{2}$$

and will refer to it as the 'effective electric field'. The Joule energy dissipation per unit volume is then $\sigma\mathscr{E}^2$. In this paper we use rationalized electromagnetic units. Equation (1*c*) will not be used in this section since it serves to generate the perturbation expansion for $\mathbf{B}$. Instead we take $\mathbf{B} = \mathbf{B}_0$.

The Joule losses will be calculated in the system of co-ordinates at rest in the sphere, where the hydrodynamic motion is steady. As discussed earlier, the velocity in the fluid is described by potential flow; thus, we write

$$\mathbf{v} = \mathbf{V}+\tfrac{1}{2}a^3\nabla(\mathbf{V}.\mathbf{r}/r^3) \tag{3}$$

relative to an origin at the centre of the sphere.

It is expedient to break up the problem into several parts and discuss first the case where $\mathbf{V}$ is parallel to $\mathbf{B}_0$. Next, the case where $\mathbf{V}$ is perpendicular to $\mathbf{B}_0$ is considered. Finally, the general case where $\mathbf{V}$ makes an oblique angle with $\mathbf{B}_0$ may be described simply in terms of the other two fundamental solutions.

(*a*) *Case* 1: $\mathbf{V}$ *parallel to* $\mathbf{B}_0$. In this case there is no induction term inside the sphere; furthermore, since $\mathbf{v}\times\mathbf{B}_0$ in the fluid has no component perpendicular to the spherical surface, it is evident that there is no surface electric charge. In fact, there is no electrostatic field at all since both $\operatorname{div}(\mathbf{v}\times\mathbf{B}_0) = 0$ and $\operatorname{div}\mathbf{j} = 0$. This implies that $\operatorname{div}\mathbf{E}_s = 0$ and, hence, no volume charge. Thus, inside the sphere the effective electric field is given by

$$\mathscr{E}' = 0, \tag{4}$$

and in the fluid by

$$\mathscr{E} = \mathbf{v}\times\mathbf{B}_0 = (3VB_0a^3/2r^3)\,[0,0,\cos\theta\sin\theta]. \tag{5}$$

Here we use spherical polar co-ordinates (r,θ,ϕ) with $\mathbf{V}$ taken as the direction of the polar axis.

The total rate of Joule energy dissipation is obtained by integrating $\sigma\mathscr{E}^2$ over the fluid volume. If this is equated to R_1V one obtains for the drag force

$$R_1 = \tfrac{2}{5}\pi a^3\sigma B_0^2V. \tag{6}$$

For this geometry there is pure drag, i.e. no deflexion, as may be readily verified by symmetry arguments.

† The conductivity of the sphere may be arbitrarily large; if the conductivity is infinite, however, the results of the calculation may have to be modified, depending on the magnitude and orientation of the magnetic field initially frozen into the sphere.

(*b*) *Case* 2: **V** *perpendicular to* $\mathbf{B}_0$. When the sphere is moving perpendicular to the magntic field, the effective electric field inside the sphere is just the electric field $\mathbf{E}'$, or

$$\mathscr{E}' = \mathbf{E}'. \tag{7}$$

In the fluid the effective electric field is given by the sum of two terms, namely

$$\mathscr{E} = \mathbf{v}\times\mathbf{B}_0+\mathbf{E}, \tag{8}$$

or

$$\mathscr{E} = -(VB_0a^3/2r^3)\left[-\sin\theta\sin\phi, 2\cos\theta\sin\phi, \cos\phi(3\cos^2\theta-1)\right]+\mathbf{V}\times\mathbf{B}_0+\mathbf{E}. \tag{8a}$$

Here **V** is taken as the polar direction and B_0 in the $\phi = 0$ plane. $\mathscr{E}$ must vanish at infinity; the second term in (8*a*) does not vanish there so it must be cancelled by a part of **E**. Now $\operatorname{div}(\sigma'\mathscr{E}') = 0$ and $\operatorname{div}(\sigma\mathscr{E}) = 0$; since the induction term in (8*a*) is itself divergence-free, we have

$$\operatorname{div}\mathbf{E} = 0, \quad \operatorname{div}\mathbf{E}' = 0.$$

This result combined with (1*b*) allows us to write formal series expansions for **E** and $\mathbf{E}'$ in terms of tesseral harmonics, $Z_j(\theta,\phi)$. Thus we write

$$\mathbf{E} = -\mathbf{V}\times\mathbf{B}-\nabla\sum_{j=0}^{\infty} b_jZ_j(\theta,\phi)\,a^{2j+1}/r^{j+1}, \tag{9}$$

$$\mathbf{E}' = -\mathbf{V}\times\mathbf{B}-\nabla\sum_{j=0}^{\infty} b'_jZ_j(\theta,\phi)\,r^j, \tag{10}$$

with the b_j and b'_j as yet undetermined constants. The boundary conditions to be satisfied are continuity of the tangential component of electric field and continuity of the normal component of current density on the spherical surface. The first of these conditions leads immediately to the result $b_j = b'_j$.

The boundary condition on the current density shows that all $b_j = 0$ except for the tesseral harmonic involving $\sin\theta\sin\phi$ for which

$$b = -\tfrac{1}{2}VB_0(\sigma+2\sigma')/(2\sigma+\sigma').$$

Incorporation of these results into the formal series expansions, (9) and (10), leads to

$$\mathbf{E}' = -\tfrac{1}{2}VB_0[3\sigma/(2\sigma+\sigma')]\,\nabla(r\sin\theta\sin\phi), \tag{11}$$

i.e. a uniform field, and

$$\mathbf{E}+\mathbf{V}\times\mathbf{B}_0 = -\tfrac{1}{2}VB_0[(\sigma+2\sigma')/(2\sigma+\sigma')]\,\nabla(a^3\sin\theta\sin\phi/r^2). \tag{12}$$

The total rate of energy dissipation is obtained by integrating $\sigma'E'^2$ over the volume of the sphere and adding the result to the integral of $\sigma\mathscr{E}^2$ over the external region. Equating this to the rate of mechanical energy dissipation, R_2V (where R_2 is the drag force), we obtain

$$R_2 = \tfrac{3}{8}\pi a^3B_0^2V\sigma(\sigma+3\sigma')/(2\sigma+\sigma'). \tag{13}$$

For this geometry there is again pure drag as may be readily verified by symmetry arguments. The current lines (lines of effective electric field) in the fluid are shown in figure 1 for the case in which $\sigma' = 0$.

(*c*) *Case* 3: *oblique angle*. When the magnetic field $\mathbf{B}_0$ makes an oblique angle with the velocity $\mathbf{V}$, the sphere experiences both deflective and drag forces. That a deflective force exists may be readily verified by considering the $\mathbf{j}\times\mathbf{B}_0$ contribution from the sphere itself. Now a vector force proportional to $B_0^2 V$ can be constructed from a linear combination of $\mathbf{B}_0\times(\mathbf{V}\times\mathbf{B}_0)$ and $\mathbf{B}_0(\mathbf{V}.\mathbf{B}_0)$ terms, or equivalently, from a combination of $B_0^2\mathbf{V}$ and $\mathbf{B}_0(\mathbf{V}.\mathbf{B}_0)$ terms. Thus, the

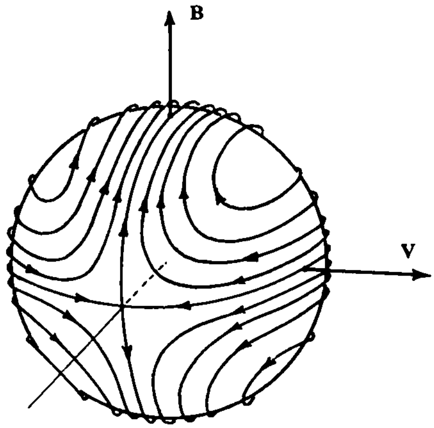

FIGURE 1. Current lines in a conducting fluid for a non-conducting sphere moving with velocity $\mathbf{V}$ at right angles to a uniform magnetic field $\mathbf{B}$.

force on the sphere (here we let $\mathbf{V}$ stand for the sphere's actual velocity in the stationary fluid) can be written

$$\mathbf{R} = -AB_0^2\mathbf{V}+C(\mathbf{V}.\mathbf{B}_0)\,\mathbf{B}_0. \tag{14}$$

The coefficients A and C may be evaluated from equations (6) and (13). We find

$$A = \tfrac{3}{5}\pi a^3[\sigma(\sigma+3\sigma')/(2\sigma+\sigma')], \tag{14a}$$

$$C = \tfrac{1}{5}\pi a^3[\sigma(7\sigma'-\sigma)/(2\sigma+\sigma')]. \tag{14b}$$

A moving sphere thus experiences, in general, a deflective force which tends to orient its motion parallel to a magnetic field line. Such forces may have a measurable effect on the orbital motion of earth satellites, although it must be emphasized that the hydrodynamic model of the fluid assumed here is quite different from that of an atmospheric plasma with its long free-path for charged particles.

2.2. *Discussion*

There are several points regarding equation (14) worth commenting upon. First, the induction drag and deflexion vanish when the conductivity of the fluid is zero. For finite σ but vanishing σ', on the other hand, there is a drag and in general a deflective force also. This result is due to the presence of complete eddy-current paths in the fluid, an example of which is shown in figure 1. Finally, it should be mentioned that a calculation of the direct electromagnetic force on the sphere, obtained from the Maxwell stress-tensor, does not give the total force $\mathbf{R}$. The integral of the normal component of the stress tensor over the surface of the sphere is equivalent in cases 1 and 2 to integrating the body force, $\mathbf{j}\times\mathbf{B}_0$, over the spherical volume; for case 1 this gives zero, while for case 2 one obtains $2\pi a^3 B_0^2 V\sigma\sigma'/(2\sigma+\sigma')$. Thus, equations (6), (13) and (14) include mechanical forces which are transmitted by the fluid. This point will be discussed further in § 3 of the paper.

2.3. *Torque on a spinning sphere from Joule losses*

The torque on a sphere spinning about its axis with angular velocity $\boldsymbol{\omega}$ in the presence of a magnetic field $\mathbf{B}_0$ can be calculated by the procedure of § 2.1. If the fluid is inviscid, there is no fluid motion and hence no induction field in the fluid. Here we shall have to make the additional restriction that the magnetic Reynolds number in the sphere, $R'_M = \sigma'\mu'\omega a^2$, is small in order that the magnetic field shall not deviate appreciably from $\mathbf{B}_0$. This is necessary because of the presence of complete eddy-current paths within the sphere itself for certain orientations.

We shall only summarize the results because the basic solutions for the electric fields are encompassed in the work of Bullard (1949) who determined the form of the electric and magnetic fields in a rotating conducting sphere of *arbitrary* magnetic Reynolds number, surrounded by a stationary shell of the same electrical conductivity. In Bullard's solutions the radial dependence is given in terms of spherical Bessel functions of the complex argument $(iR'_M)^{\frac{1}{2}}r/a$, but by making the appropriate small argument expansion of the Bessel functions, his solutions reduce to ours in lowest order.

(*a*) *Case* 4: $\boldsymbol{\omega}$ *parallel to* $\mathbf{B}_0$. The induction field inside the sphere may be written as

$$\mathbf{E}_{\text{induc}} = \tfrac{1}{2}\omega B_0 \nabla(r^2\sin^2\theta), \tag{15}$$

where, for convenience, $\boldsymbol{\omega}$ is taken as the direction of the polar axis. Although the divergence of this field does not vanish, its curl does. Thus the divergence may be cancelled by an appropriate electrostatic field. For steady conduction the effective electric field must be divergence-free. We find that all boundary conditions can be satisfied by taking the effective electric field inside the sphere to be

$$\mathscr{E}' = -[\tfrac{1}{2}\sigma\omega B_0/(3\sigma+2\sigma')]\,\nabla[r^2(3\cos^2\theta-1)], \tag{16}$$

and the electric field in the fluid to be

$$\mathbf{E} = [\sigma'\omega B_0/(3\sigma+2\sigma')]\,\nabla[(3\cos^2\theta-1)\,a^5/3r^3]. \tag{17}$$

By equating the rate of Joule energy dissipation to $D_1\omega$, we obtain, for the drag torque,

$$D_1 = \tfrac{8}{15}\pi a^5\omega B_0^2\sigma\sigma'/(3\sigma+2\sigma'). \tag{18}$$

If we let $\sigma = \sigma'$, this result reduces to that found by Bullard (see, for example, his equation (29)].

(*b*) *Case* 5: $\boldsymbol{\omega}$ *perpendicular to* $\mathbf{B}_0$. When $\boldsymbol{\omega}$ is oriented perpendicular to $\mathbf{B}_0$, the effective electric field inside the sphere may be written as

$$\mathscr{E}' = -\boldsymbol{\omega}B_0 r\sin\theta\cos\phi + \mathbf{E}' \tag{19}$$

if $\boldsymbol{\omega}$ is taken as the polar direction and $\mathbf{B}_0$ is in the $\phi = 0$ plane. In the fluid,

$$\mathscr{E} = \mathbf{E}. \tag{20}$$

The boundary conditions are satisfied by taking

$$\mathbf{E}' = [\sigma'\omega B_0/(3\sigma + 2\sigma')]\,\nabla(r^2 \cos\theta \sin\theta \cos\phi) \tag{21}$$

and

$$\mathbf{E} = [\sigma'\omega B_0 a^5/(3\sigma + 2\sigma')]\,\nabla(\cos\theta \sin\theta \cos\phi/r^3). \tag{22}$$

From the energy dissipation, we readily obtain

$$D_2 = \tfrac{4}{5}\pi a^5 \omega B_0^2 \sigma'(\sigma + \tfrac{1}{3}\sigma')/(3\sigma + 2\sigma'). \tag{23}$$

In this case there is drag torque on a conducting sphere even when the conductivity σ of the fluid vanishes. This results from eddy currents wholly inside the sphere. The torque can be calculated alternatively by integrating $\mathbf{r} \times (\mathbf{j} \times \mathbf{B}_0)$ over the sphere; in both cases, (4) and (5), the total torque is in the direction $-\boldsymbol{\omega}$.

(*c*) *Case* 6: *oblique angle*. By analogy with case 3, the vector torque on a spinning sphere may be written as

$$\mathbf{D} = -\alpha B_0^2 \boldsymbol{\omega} + \gamma \mathbf{B}_0(\mathbf{B}_0 \cdot \boldsymbol{\omega}). \tag{24}$$

Comparison with equations (20) and (25) yields the relations

$$\alpha = \tfrac{4}{15}\pi a^5 \sigma'(3\sigma + \sigma')/(3\sigma + 2\sigma'), \tag{24a}$$

$$\gamma = \tfrac{4}{15}\pi a^5 \sigma'(\sigma + \sigma')/(3\sigma + 2\sigma'). \tag{24b}$$

3. More detailed mathematical analysis

In calculating the resistive force on a translating sphere by the Joule-loss method, we found that part of the force had to be transmitted via mechanical forces in the fluid. It is of interest, therefore, to see what can be said about the perturbed hydrodynamic flow.

3.1. *Fundamental equations*

We shall work in the system of co-ordinates in which the sphere is at rest, and look for a steady solution to the hydrodynamic equations. The equations governing the steady motion of an incompressible, inviscid, electrically conducting fluid are

$$\left.\begin{aligned} &(a)\ \operatorname{div}\mathbf{v} = 0,\\ &(b)\ \operatorname{curl}\mathbf{v}\times\mathbf{v} = -\nabla[(p/\rho) + \tfrac{1}{2}v^2] + (1/\rho\mu)\operatorname{curl}\mathbf{B}\times\mathbf{B}, \end{aligned}\right\} \tag{25}$$

together with the equations for the electromagnetic field, equations (1). The velocity, pressure, and magnetic field can be developed in perturbation series in powers of an expansion parameter proportional to B_0^2/V^2 (Ludford & Murray 1960). The zero-order magnetic field can also be expanded in powers of the magnetic Reynolds number, R_M (see § 3.2).

Writing $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1$, with

$$\mathbf{v}_0 = \mathbf{V} + \tfrac{1}{2}a^3 \nabla(\mathbf{V}\cdot\mathbf{r}/r^3) \tag{26}$$

the unperturbed velocity, and $\mathbf{v}_1$ the perturbed velocity of the fluid, writing the pressure $p = p_0 + p_1$, and $\mathbf{B} = \mathbf{B}(0) + \mathbf{B}_1$, we obtain the linearized equations for $\mathbf{v}_1$ (here we have written $\mathbf{B}(0)$ for the zero-order field since $\mathbf{B}_0$ has been used to denote the uniform field) in the form

$$\left.\begin{aligned} &(a)\ \operatorname{div}\mathbf{v}_1 = 0, \\ &(b)\ (\operatorname{curl}\mathbf{v}_1)\times\mathbf{v}_0 = -\nabla(\mathbf{v}_0\cdot\mathbf{v}_1+p_1/\rho)+(1/\rho\mu)\operatorname{curl}\mathbf{B}(0)\times\mathbf{B}(0).\end{aligned}\right\} \tag{27}$$

If equation (27 *b*) is put in dimensionless form by writing all velocities in units of V, all pressures in units of ρV^2, and all distances in units of a, then it is seen that the magnetic body force is of order

$$\xi = \beta R_M, \tag{28}$$

where
$$\beta = B_0^2/\mu\rho V^2, \quad R_M = \sigma\mu V a.$$

Not only must ξ be small in order to treat $\mathbf{v}_1$ as a perturbation, but the magnetic Reynolds number R_M must also be small in order that the magnetic field shall not deviate appreciably from $\mathbf{B}_0$ in the current-carrying region.

If the sphere's velocity $\mathbf{V}$ is too small, then (28) shows that the magnetic force cannot be treated as a perturbation. Thus it is not immediately clear that a solution to equation (27) is appropriate when the sphere is brought to a steady velocity after starting from rest. However, when the sphere is moving very slowly the effect of viscosity cannot be neglected. Chester (1957) has studied the effect of a magnetic field on Stokes's flow in a conducting fluid and has found a steady solution of the hydrodynamic equations which does not diverge as $\mathbf{V}$ approaches zero.

3.2. *Expansion of* $\mathbf{B}(0)$ *in terms of the magnetic Reynolds number*

The use in § 1 of the uniform field $\mathbf{B}_0$ for the zero-order field (with reference to an expansion in powers of R_M) requires some justification, particularly since Ludford & Murray found that such a perturbation expansion is irregular. The magnetic field $\mathbf{B}(0)$, or for brevity $\mathbf{B}$, in the fluid must satisfy the equations

$$\operatorname{curl}\mathbf{B} = \sigma\mu[\mathbf{E}+\mathbf{v}_0\times\mathbf{B}], \tag{29}$$

$$\operatorname{div}\mathbf{B} = 0. \tag{30}$$

We consider only case 1 geometry. Here there is no electric field. For the remainder of the paper we shall make $\mathbf{v}$, $\mathbf{r}$, and $\mathbf{B}$ dimensionless by referring them to the velocity at infinity $\mathbf{V}$, the radius of the sphere a, and the uniform magnetic field B_0, respectively. Let us write

$$\mathbf{B} = \mathbf{v}_0+\mathbf{b}. \tag{31}$$

The first term, $\mathbf{v}_0$, satisfies both equations (29) and (30) and gives the correct asymptotic form to the field at infinity. Thus we need consider only $\mathbf{b}$. If we satisfy (30) by means of

$$b_r = \frac{1}{r^2\sin\theta}\frac{\partial A}{\partial\theta}, \quad b_\theta = -\frac{1}{r\sin\theta}\frac{\partial A}{\partial r}, \quad (b_\phi = 0), \tag{32}$$

with $A = A(r,\theta)$, then in the fluid, (29) becomes

$$\frac{\partial^2 A}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial A}{\partial\theta}\right) = R_M\left[(1-r^{-3})\cos\theta\frac{\partial A}{\partial r}-\frac{1}{r}(1+\tfrac{1}{2}r^{-3})\sin\theta\frac{\partial A}{\partial\theta}\right]. \tag{33}$$

For small values of R_M, the appropriate solution to (33) may be found by a perturbation expansion. We require a solution with dipole character to cancel approximately the dipole behaviour in $\mathbf{v}_0$. This is just the solution, which has been discussed by Ludford & Murray (1960),

$$A = \exp[-\tfrac{1}{2}R_M r(1-\cos\theta)]\{(\kappa/r)\sin^2\theta + R_M[\tfrac{1}{2}\kappa + (\lambda/r^2)\cos\theta]\sin^2\theta\} + O(R_M^2). \tag{34}$$

Inside the sphere, the total magnetic field may be derived from

$$A' = (kr^2 + R_M lr^3\cos\theta)\sin^2\theta. \tag{35}$$

Continuity of the normal component of $\mathbf{B}$ and the tangential component of magnetic intensity across the sphere boundary gives

$$\kappa = k = \frac{3\mu'}{2(\mu'+2\mu)}, \quad \lambda = -\frac{9\mu\mu'}{4(\mu'+2\mu)(3\mu+2\mu')}, \quad l = -\frac{2\mu'}{3\mu}\lambda,$$

where μ and μ' are the permeabilities of fluid and sphere, respectively. For our problem $\mu = \mu'$; thus

$$\kappa = k = \tfrac{1}{2}, \quad \lambda = -\tfrac{3}{20}, \quad l = \tfrac{1}{10}.$$

To zero order in R_M, the magnetic field is given by

$$\mathbf{B} = \mathbf{v}_0 + \exp[-\tfrac{1}{2}R_M r(1-\cos\theta)]\left\{\frac{\cos\theta}{r^3}, \frac{\sin\theta}{2r^3}, 0\right\}, \tag{36}$$

and $\mathbf{v}_0 \times \mathbf{B}$ is found to be

$$\mathbf{v}_0 \times \mathbf{B} = (\tfrac{3}{2}r^{-3})\exp[-\tfrac{1}{2}R_M r(1-\cos\theta)]\{0, 0, \sin\theta\cos\theta\}.$$

The Joule-loss density is thus

$$\sigma\mathscr{E}^2 = \tfrac{9}{4}\rho V^3 a^2 \beta R_M r^{-6}\cos^2\theta\sin^2\theta\exp[-R_M r(1-\cos\theta)],$$

which differs from the result found in § 2.1 only by the presence of the exponential factor. When this loss density is integrated over the volume of the fluid, the total Joule loss is the same as before to the lowest order, $[O(R_M)]$.

3.3. *Drag force from the perturbed velocity* (*case* 1)

We have not been able to establish the existence of a solution to equation (27) for each of the cases discussed in § 2.1. But Ludford & Murray have discussed the method of solution for equation (27) when the magnetic field $\mathbf{B}$ has axial symmetry about the direction $\mathbf{V}$ and hence their results may be applied directly to our case 1. The essence of their method is to take the curl of (27*b*), giving

$$\operatorname{curl}[\boldsymbol{\omega} \times \mathbf{v}_0] = \beta R_M \operatorname{curl}[(\mathbf{v}_0 \times \mathbf{B}) \times \mathbf{B}], \tag{37}$$

where $\boldsymbol{\omega} = \operatorname{curl}\mathbf{v}_1$ is the vorticity. Since this equation and $\boldsymbol{\omega}$ each have but a single component for case 1 geometry, the resulting equation for ω is relatively simple and may be solved by means of a series expansion,†

$$\omega = \sum_{n=1}^{\infty} \omega_n(r)\sin\theta P_n'(\cos\theta).$$

† For the necessity of this form of expansion the reader is referred to Ludford & Murray, p. 524.

Ludford & Murray show for this geometry that the drag force due to the pressure is $R = \rho V^2 a^2 D_p$, where

$$D_p = \pi\beta \int_1^\infty \int_0^\pi (F_\theta \sin\theta + 2F_r \cos\theta) \sin\theta \, d\theta \frac{dr}{r}. \tag{38}$$

Here $\mathbf{F} = \{F_r, F_\theta, 0\}$ is defined as $\mathbf{F} = R_M(\mathbf{v}_0 \times \mathbf{B}) \times \mathbf{B}$. Using equation (36) for the magnetic field, it is readily seen that

$$F_\theta \sin\theta + 2F_r \cos\theta = \tfrac{9}{2} R_M \exp[-\tfrac{1}{2} R_M r(1-\cos\theta)] r^{-3} \sin^2\theta \cos^2\theta + O(R_M^2).$$

In evaluating D_p we may set $R_M = 0$ in this last exponential since this leads only to an error of $O(R_M^2)$. The result is

$$D_p = \tfrac{3}{5}\pi\beta R_M, \tag{38a}$$

which agrees with (6).

In our problem (case 1), this is the total drag force since the Maxwell stress-tensor gives no contribution to this order.

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